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Exact and asymptotic stability analyses of a coated elastic half-space

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Abstract

We study the buckling of a pre-stressed coated elastic half-space with the aid of the exact theory of nonlinear elasticity, treating the coating as an elastic layer and using its thickness as a small parameter. Two asymptotic limits are identified: $\mathscr{A}_{jilk} = O(kh\mathscr{A}_{jilk})$ and $\mathscr{A}_{jilk} = O(k^3h^3\mathscr{A}_{jilk})$, where $\mathscr{A}_{jilk}$ and $\mathscr{A}_{jilk}$ are the elastic moduli for the half-space and the coating, respectively, k is a mode number and h the thickness of coating. The first limit corresponds to the case when the coating and half-space exert maximum effect on each other and the second limit corresponds to the classical model equation for a plate supported by an elastic foundation. For each limit the leading order bifurcation condition is derived using two different methods. In the first method we derive the leading order governing equations first and then obtain from them the bifurcation condition. In the second method we derive the exact bifurcation condition first and then take the thin-layer limit. The two methods are found to yield the same results, assuring us that the leading order governing equations are asymptotically consistent. These leading order governing equations in the thin-layer limit are then compared with those assumed or derived by previous researchers. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

A coated elastic half-space can be used to model the structures in a number of engineering situations, such as surface processing and thin-film deposition. It can also be used as a model for thick plates reinforced with high strength thin layers and plates on elastic foundations. The coating in such situations is usually very thin and it is desirable to make use of this fact and develop a simple, but rational, theory to describe its behaviour. Gurtin and Murdoch (1975) presented a general nonlinear model of surface-stressed solids in the setting of modern finite elasticity. Recently, Steigmann and Ogden

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(1997) generalized the Gurtin–Murdoch theory by incorporating flexural stiffness directly into the constitutive response of the coating. They also linearized their nonlinear governing equations and conducted a linear stability analysis. Different approximate linear theories have also been used by Gille and Rau (1984), Yu et al. (1991) and Shield et al. (1994) to investigate the linear stability properties of a coated elastic half-space. In all these studies, simplification results from the fact that effect of the coating is incorporated into the interfacial continuity conditions.

In this paper we assume that a coated elastic half-space can be viewed as the limit of an elastic layer bonded to an elastic half-space as the layer thickness becomes vanishingly small. We aim to derive the appropriate asymptotic interfacial conditions and compare them with those used by previous researchers.

This paper is divided into four sections as follows. In the next section, we first write down the exact linearized governing equations, boundary and interfacial conditions for a layer bonded to an elastic half-space. The layer is assumed to have arbitrary thickness, and the layer and half-space are assumed to be subjected to a uni-axial compression. These equations have, as dependent variables, the incremental displacement components (the incremental displacement is the difference between the positions of a material particle in the uniformly stressed configuration and another possibly buckled configuration). Solving these equations following the standard procedure of linear stability analysis yields an exact bifurcation condition which relates the critical principal stretch to kh, where k is the mode number and h is the layer thickness. Such a linear stability analysis has previously been conducted using the exact theory of finite elasticity by Dorris and Nemat-Nasser (1980), Ogden and Sotiropoulos (1996) and Bigoni et al. (1997). Whereas these previous studies are mainly concerned with the numerical solution of the exact bifurcation condition, our emphasis in this paper will be on analyzing the asymptotic structure of the exact bifurcation condition. In section 3, we derive the leading order governing equations for the two asymptotic limits referred to in the abstract. We show that these leading order governing equations yield the same bifurcation condition as that obtained by taking the thin-layer limit in the exact bifurcation condition. In the final section we compare our leading order governing equations with those assumed or derived by previous researchers.

2. Exact theory

2.1. Exact bifurcation condition

In this section we present the governing equations for a layer bonded to an elastic half-space and derive the exact bifurcation condition. The exact bifurcation condition has been derived recently by Ogden and Sotiropoulos (1996). Our attention will be focused on the thin layer limit. Both the layer and half-space are assumed to be isotropic and compressible in their unstressed configurations. A certain finite deformation is applied to this bonded structure, giving rise to the possibility that the structure may buckle. We choose a co-ordinate system in which the origin is located at the interface in the stressed configuration, the x_1- and x_3-axes lie in the interface and aligned with two of the three principal axes of stretch, and the x_2-axis perpendicular to the interface and pointing into the layer. Thus a representative material particle in the stressed configuration has co-ordinates (x_i). As the bonded structure buckles, this same material particle moves to a new point with co-ordinates $\tilde{x}_i$ and we may write $\tilde{x}_i = \tilde{x}_i(x_j)$, and define the incremental displacement components u_i as

$$u_i = \tilde{x}_i - x_i.$$

We assume that the incremental deformation is plane strain and is of small amplitude. It can then be shown (see e.g. Dowaikh and Ogden 1991) that the linearized governing equations are given by

$$\mathscr{A}_{jilk}u_{k,lj} = 0, \tag{1}$$

and the incremental traction on a surface with unit normal (n_j) is given by

$$T_i = \mathscr{A}_{jilk}u_{k,l}n_j, \tag{2}$$

where $\mathscr{A}_{jilk}$ are the first order instantaneous moduli and their non-zero components are given by

$$J\mathscr{A}_{iijj} = \lambda_i\lambda_j W_{ij}$$

$$J\mathscr{A}_{ijij} = \begin{cases} (\lambda_i W_i - \lambda_j W_j)\lambda_i^2/(\lambda_i^2 - \lambda_j^2) & i \neq j, \quad \lambda_i \neq \lambda_j \\ \frac{1}{2}(J\mathscr{A}_{iiii} - J\mathscr{A}_{iijj} + \lambda_i W_i) & i \neq j, \quad \lambda_i = \lambda_j \end{cases}$$

$$\mathscr{A}_{ijji} = \mathscr{A}_{jiij} = \mathscr{A}_{ijij} - \sigma_i. \tag{3}$$

In these expressions λ_i and σ_i are the principal stretches and stresses associated with the finite deformation, $J = \lambda_1\lambda_2\lambda_3$, W is the strain energy function and $W_i = \partial W/\partial\lambda_i$, $W_{ij} = \partial^2 W/\partial\lambda_i\partial\lambda_j$, $\sigma_i = J^{-1}\lambda_i W_i$. As in Steigmann and Ogden (1997) we define the notation

$$\alpha_{ij} = \mathscr{A}_{iijj} \quad i, j \in \{1, 2\},$$

$$\gamma_{ij} = \mathscr{A}_{ijij} \quad i \neq j,$$

$$\delta_{12} = \delta_{21} = \alpha_{12} + \gamma_{12} - \sigma_1 = \alpha_{21} + \gamma_{21} - \sigma_2,$$

$$2\beta_{12} = \alpha_{11}\alpha_{22} + \gamma_{12}\gamma_{21} - \delta_{12}^2. \tag{4}$$

To fix ideas, we shall assume that the pre-stress takes the form of a uni-axial compression along the x_1-direction so that $\sigma_2 = 0$, but most of our following analysis is valid for general pre-stress. We also assume that the pre-stressed state is a state of plane strain so that $\lambda_3 = 1$. The principal stretch λ_2 can then be expressed in terms of λ_1 with the aid of the equation obtained from $\sigma_2 = 0$. To simplify notation we shall write λ_1 as λ.

Eqs. (1)–(4) are valid for both the layer and the half-space. From now on we shall adopt the convention whereby a symbol with an over-bar is associated with the layer, whereas the same symbol without an over-bar is associated with the half-space. Thus, for instance, $\mathscr{A}_{jilk}$ and $\bar{\mathscr{A}}_{jilk}$ are the elastic moduli for the half-space and the layer, respectively, and the counterparts of (1) and (2) for the layer are

$$\bar{\mathscr{A}}_{jilk}\bar{u}_{k,lj} = 0, \quad \bar{T}_i = \bar{\mathscr{A}}_{jilk}\bar{u}_{k,l}n_j. \tag{5}$$

We look for a buckling solution of the form

$$\bar{u}_j = \bar{H}_j(kx_2)\mathrm{e}^{ikx_1}, \quad u_j = H_j(kx_2)\mathrm{e}^{ikx_1}, \quad j = 1, 2 \tag{6}$$

where k is the mode number, and observing the convention made above, $(\bar{u}_j, \bar{H}_j)$ and (u_j, H_j) are associated with the layer and half-space, respectively. On substituting (6a) into (5a), we obtain two second order differential equations for $\bar{H}_1$ and $\bar{H}_2$. Solving these equations yields

$$\bar{H}_1(kx_2) = \sum_{j=1}^{4} \bar{A}_j \exp(k\bar{p}_j x_2), \quad \bar{H}_2(kx_2) = \sum_{j=1}^{4} \frac{i\bar{p}_j \bar{\delta}_{12}}{\bar{\gamma}_{12} - \bar{\alpha}_{22}\bar{p}_j^2} \bar{A}_j \exp(k\bar{p}_j x_2), \tag{7}$$

where $\bar{A}_j$ are disposable constants, $\bar{p}_j$ are the four roots of

$$\bar{\alpha}_{22}\bar{\gamma}_{21}\bar{p}^4 - 2\bar{\beta}_{12}\bar{p}^2 + \bar{\gamma}_{12}\bar{\alpha}_{11} = 0, \tag{8}$$

and are ordered such that $\bar{p}_1$ and $\bar{p}_2$ have positive real parts and $\bar{p}_3 = -\bar{p}_1$, $\bar{p}_4 = -\bar{p}_2$.

The counter-part of (7) for the half-space is

$$H_1(kx_2) = \sum_{j=1}^{2} A_j \exp(kp_j x_2), \quad H_2(kx_2) = \sum_{j=1}^{2} \frac{ip_j \delta_{12}}{\gamma_{12} - \alpha_{22}p_j^2} A_j \exp(kp_j x_2), \tag{9}$$

where A_1, A_2 are disposable constants and the terms involving p_3, p_4 are neglected to satisfy the decay conditions $u_1, u_2 \to 0$ as $x_2 \to -\infty$.

The six constants $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$, $\bar{A}_4$, A_1, A_2 are determined by the boundary conditions

$$\bar{T}_1 = 0, \quad \bar{T}_2 = 0, \quad \text{on } x_2 = h \tag{10}$$

and the continuity conditions at the interface:

$$\bar{T}_1 = T_1, \quad \bar{T}_2 = T_2, \quad \bar{u}_1 = u_1, \quad \bar{u}_2 = u_2, \quad \text{on } x_2 = 0. \tag{11}$$

On substituting (2), (5b), (6), (7) and (9) into these conditions, we obtain six linear homogeneous equations for the six constants. The four equations corresponding to (11) can be solved to express $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$, $\bar{A}_4$ in terms of A_1 and A_2 as

$$\bar{A}_1 = \frac{1}{2}[(d_1 + d_3)A_1 + (d_2 + d_4)A_2],$$

$$\bar{A}_2 = \frac{1}{2}[(e_1 + e_3)A_1 + (e_2 + e_4)A_2],$$

$$\bar{A}_3 = \frac{1}{2}[(d_1 - d_3)A_1 + (d_2 - d_4)A_2],$$

$$\bar{A}_4 = \frac{1}{2}[(e_1 - e_3)A_1 + (e_2 - e_4)A_2] \tag{12}$$

where

$$d_1 = \frac{\bar{b}_2 - b_1}{\bar{b}_2 - \bar{b}_1}, \quad d_2 = \frac{\bar{b}_2 - b_2}{\bar{b}_2 - \bar{b}_1},$$

$$e_1 = \frac{b_1 - \bar{b}_1}{\bar{b}_2 - \bar{b}_1}, \quad e_2 = \frac{b_2 - \bar{b}_1}{\bar{b}_2 - \bar{b}_1},$$

$$d_3 = \frac{a_1\bar{c}_2 - \bar{a}_2 c_1}{\bar{a}_1\bar{c}_2 - \bar{a}_2\bar{c}_1}, \quad d_4 = \frac{a_2\bar{c}_2 - \bar{a}_2 c_2}{\bar{a}_1\bar{c}_2 - \bar{a}_2\bar{c}_1},$$

$$e_3 = \frac{\bar{a}_1 c_1 - a_1\bar{c}_1}{\bar{a}_1\bar{c}_2 - \bar{a}_2\bar{c}_1}, \quad e_4 = \frac{\bar{a}_1 c_2 - a_2\bar{c}_1}{\bar{a}_1\bar{c}_2 - \bar{a}_2\bar{c}_1},$$

and the constants a_j, b_j and c_j are defined by

$$a_j = (\alpha_{11} - \gamma_{21} p_j^2)/(\delta_{12} p_j), \quad b_j = \alpha_{12} - \alpha_{22} p_j a_j, \quad c_j = \gamma_{21}(p_j + a_j)$$

with no summation on the repeated suffix j. The expressions for $\bar{a}_j$, $\bar{b}_j$ and $\bar{c}_j$ have the same form except that an over-bar is added to each of the symbols appearing on the right hand sides.

On substituting (12) into the two equations corresponding to (10), we obtain

$$f_{11}A_1 + f_{12}A_2 = 0, \quad f_{21}A_1 + f_{22}A_2 = 0, \tag{13}$$

where

$$f_{11} = \bar{b}_1(d_1 \text{ ch}\zeta_1 + d_3 \text{ sh}\zeta_1) + \bar{b}_2(e_1 \text{ ch}\zeta_2 + e_3 \text{ sh}\zeta_2),$$

$$f_{12} = \bar{b}_1(d_2 \text{ ch}\zeta_1 + d_4 \text{ sh}\zeta_1) + \bar{b}_2(e_2 \text{ ch}\zeta_2 + e_4 \text{ sh}\zeta_2),$$

$$f_{21} = \bar{c}_1(d_1 \text{ sh}\zeta_1 + d_3 \text{ ch}\zeta_1) + \bar{c}_2(e_1 \text{ sh}\zeta_2 + e_3 \text{ ch}\zeta_2),$$

$$f_{22} = \bar{c}_1(d_2 \text{ sh}\zeta_1 + d_4 \text{ ch}\zeta_1) + \bar{c}_2(e_2 \text{ sh}\zeta_2 + e_4 \text{ ch}\zeta_2), \tag{14}$$

and $\zeta_1 = kh\bar{p}_1$, $\zeta_2 = kh\bar{p}_2$, the ch and sh standing for cosh and sinh, respectively.

The bifurcation condition is then given by $f_{11}f_{22} - f_{12}f_{21} = 0$. With the use of (14), this condition can be manipulated into the form

$$b_1c_2 - b_2c_1 + g_1(\text{ch}\zeta_1 \text{ ch}\zeta_2 - 1) + g_2 \text{ ch}\zeta_1 \text{ sh}\zeta_2 + g_3 \text{ ch}\zeta_2 \text{ sh}\zeta_1 + g_4 \text{ sh}\zeta_1 \text{ sh}\zeta_2 = 0, \tag{15}$$

where

$$g_1 = \bar{b}_1\bar{c}_2(d_1e_4 - d_2e_3) + \bar{b}_2\bar{c}_1(d_4e_1 - d_3e_2),$$

$$g_2 = \bar{b}_1\bar{c}_2(d_1e_2 - d_2e_1) + \bar{b}_2\bar{c}_1(d_4e_3 - d_3e_4),$$

$$g_3 = \bar{b}_1\bar{c}_2(d_3e_4 - d_4e_3) + \bar{b}_2\bar{c}_1(d_2e_1 - d_1e_2),$$

$$g_4 = \bar{b}_1\bar{c}_2(d_3e_2 - d_4e_1) + \bar{b}_2\bar{c}_1(d_2e_3 - d_1e_4). \tag{16}$$

Eq. (15) is the general and exact bifurcation condition for a layer of arbitrary thickness bonded to a half-space. In the case when the pre-stress takes the form of a uni-axial compression, Eq. (15) can be solved numerically to yield the critical stretch λ as a function of kh.

2.2. Bifurcation condition for a half-space

The bifurcation condition for a half-space can be recovered from (15) in two different ways. First, we may take the layer and half-space to be composed of the same material. We then have $a_j=\bar{a}_j$, $b_j=\bar{b}_j$, $c_j=\bar{c}_j$, $(j = 1, 2)$, $d_1=d_3=1$, $d_2=d_4=0$, $e_1=e_3=0$, $e_2=e_4=1$. It follows from (16) that $g_1=g_2=g_3=g_4=b_1c_2-b_2c_1$. After some manipulations, the general bifurcation condition is reduced to

$$(b_1c_2 - b_2c_1)\mathrm{e}^{\zeta_1+\zeta_2} \equiv \Pi_0\Pi\mathrm{e}^{\zeta_1+\zeta_2} = 0, \tag{17}$$

where

$$\Pi = \sqrt{\alpha_{11}\alpha_{22}\gamma_{12}\gamma_{21}}(\sqrt{\alpha_{11}\alpha_{22}} + \sqrt{\gamma_{12}\gamma_{21}}) - \sqrt{\gamma_{12}\gamma_{21}}\alpha_{12}^2 - \sqrt{\alpha_{11}\alpha_{22}}(\delta_{12} - \alpha_{12})^2,$$

$$\Pi_0 = -\sqrt{\frac{\gamma_{21}}{\gamma_{12}}}\frac{p_2 - p_1}{\delta_{12}}. \tag{18}$$

Although $p_1-p_2=0$ satisfies (17), it is well-known that it in fact corresponds to the trivial solution. It then follows from (17) that the bifurcation condition for a pre-stressed half-space is given by

$$\Pi = 0, \tag{19}$$

which is well-known in the literature, see e.g. Dowaikh and Ogden (1991).

Alternatively, a half-space can be recovered by taking $h = 0$. The general bifurcation condition (15) in this limit reduces to $b_1c_2-b_2c_1=0$ which is clearly equivalent to (17).

2.3. Bifurcation condition in the thin-layer limit

Our main concern in this paper is with the stability of a coated elastic half-space. We treat a coated half-space as the thin-layer limit of the bonded structure which we have just considered. Since the half-space does not have a natural lengthscale, the term 'thin layer' only makes sense when the thickness of the layer is compared with a reference lengthscale. For the present buckling problem, a natural choice for the reference lengthscale is $1/k$ which is the wavelength of the bifurcation mode divided by 2π. Thus the thin layer limit corresponds to

$$\epsilon \to 0, \quad \text{where} \quad \epsilon = kh.$$

For ϵ small, the general bifurcation condition (15) can be expanded into the following Taylor series:

$$b_1c_2 - b_2c_1 + \omega_1\epsilon + \frac{1}{2}\omega_2\epsilon^2 + \frac{1}{6}\omega_3\epsilon^3 + \frac{1}{24}\omega_4\epsilon^4 + \cdots = 0, \tag{20}$$

where

$$\omega_1 = g_2\bar{p}_2 + g_3\bar{p}_1,$$

$$\omega_2 = g_1(\bar{p}_1^2 + \bar{p}_2^2) + 2g_4\bar{p}_1\bar{p}_2,$$

$$\omega_3 = g_2(\bar{p}_2^3 + 3\bar{p}_1^2\bar{p}_2) + g_3(\bar{p}_1^3 + 3\bar{p}_1\bar{p}_2^2),$$

$$\omega_4 = g_1(\bar{p}_1^4 + \bar{p}_2^4 + 6\bar{p}_1^2\bar{p}_2^2) + 4g_4\bar{p}_1\bar{p}_2(\bar{p}_1^2 + \bar{p}_2^2). \tag{21}$$

On substituting the relevant expressions into the right hand sides of (21) and simplifying, we find, purely in terms of material constants,

$$\omega_1 = \Pi_0(s_1\sqrt{\alpha_{11}\gamma_{21}} + s_2\sqrt{\alpha_{22}\gamma_{12}})\sqrt{\Delta},$$

$$\omega_3 = \Pi_0(s_3\sqrt{\alpha_{11}\gamma_{21}} + s_4\sqrt{\alpha_{22}\gamma_{12}})\sqrt{\Delta},$$

$$\omega_2 = \Pi_0[s_5(\sqrt{\gamma_{12}\gamma_{21}} + \sqrt{\alpha_{11}\alpha_{22}}) + s_6(\gamma_{21}\sqrt{\alpha_{11}\alpha_{22}} - \alpha_{12}\sqrt{\gamma_{12}\gamma_{21}}) + s_7\Pi],$$

$$\omega_4 = \Pi_0[s_8(\sqrt{\gamma_{12}\gamma_{21}} + \sqrt{\alpha_{11}\alpha_{22}}) + s_9(\gamma_{21}\sqrt{\alpha_{11}\alpha_{22}} - \alpha_{12}\sqrt{\gamma_{12}\gamma_{21}}) + s_{10}\Pi], \tag{22}$$

where

$$\Delta = 2\beta_{12} + 2\sqrt{\alpha_{11}\alpha_{22}\gamma_{12}\gamma_{21}},$$

$$s_1 = \bar{\gamma}_{12} - \bar{\gamma}_{21},$$

$$s_2 = \bar{\alpha}_{11} - \bar{\alpha}_{12}^2/\bar{\alpha}_{22},$$

$$s_3 = \frac{2}{\bar{\alpha}_{22}\bar{\gamma}_{21}}(\bar{\beta}_{12}s_1 + \bar{\alpha}_{22}\bar{\gamma}_{12}s_2),$$

$$s_4 = \frac{2}{\bar{\alpha}_{22}\bar{\gamma}_{21}}(\bar{\beta}_{12}s_2 + \bar{\alpha}_{11}\bar{\gamma}_{21}s_1),$$

$$s_5 = 2s_1s_2,$$

$$s_6 = 2\frac{\bar{\alpha}_{11}\bar{\alpha}_{22} - \bar{\alpha}_{12}^2 + \bar{\alpha}_{12}(\bar{\gamma}_{21} - \bar{\gamma}_{12})}{\bar{\alpha}_{22}}, \tag{23}$$

$$s_7 = \frac{2(\bar{\beta}_{12} + \bar{\alpha}_{12}\bar{\gamma}_{21})}{\bar{\alpha}_{22}\bar{\gamma}_{21}},$$

$$s_8 = s_1s_4 + s_2s_3,$$

$$s_9 = 4\frac{(\bar{\beta}_{12} - \bar{\alpha}_{12}\bar{\gamma}_{12})(\bar{\alpha}_{11}\bar{\alpha}_{22} - \bar{\alpha}_{12}^2) + (\bar{\beta}_{12}\bar{\alpha}_{12} - \bar{\alpha}_{11}\bar{\alpha}_{22}\bar{\gamma}_{21})(\bar{\gamma}_{21} - \bar{\gamma}_{12})}{\bar{\gamma}_{21}\bar{\alpha}_{22}^2},$$

$$s_{10} = \frac{2}{\bar{\alpha}_{22}^2 \bar{\gamma}_{21}^2} \{ s_1(\bar{\beta}_{12} + \bar{\alpha}_{11}\bar{\alpha}_{22})\bar{\gamma}_{21} + s_2(\bar{\beta}_{12} + \bar{\gamma}_{12}\bar{\gamma}_{21})\bar{\alpha}_{22} \}. \tag{24}$$

It can be seen from (20) that for the coating and half-space to exert the same order of effect on each other, the only possibility is for the first two terms $b_1c_2 - b_2c_1$ (which is independent of kh) to be balanced by the other kh-dependent terms. Since ϵ is small, one of the ω_i's must be large. An inspection of the expressions for ω_1 ω_2, ω_3 and ω_4 shows that this in turn implies that $\mathscr{A}_{jilk}$ must be much greater than $\mathscr{A}_{jilk}$. This means that the coating must be much stiffer than the half-space, as one would intuitively expect. Without loss of generality, we may assume that $\mathscr{A}_{jilk} = O(1)$. If we introduce the notation

$$\alpha = \max(\bar{\mathscr{A}}_{jilk}, j, i, l, k \in \{1, 2\}),$$

then for α large Eq. (20) takes the form

$$O(1) + O(\alpha)\epsilon + O(\alpha^2)\epsilon^2 + O(\alpha)\epsilon^3 + O(\alpha^2)\epsilon^4 + \cdots = 0. \tag{25}$$

Clearly as α is increased from $O(1)$, a balance is first achieved when

$$\alpha = O(1/\epsilon), \tag{26}$$

and it is the second and third terms that balances the first term. After some manipulations, we find that when (26) holds, Eq. (20) to leading order reduces to

$$\Pi + (s_1^*\sqrt{\alpha_{11}\gamma_{21}} + s_2^*\sqrt{\alpha_{22}\gamma_{12}})\sqrt{\Delta} + s_1^* s_2^*(\sqrt{\alpha_{11}\alpha_{22}} + \sqrt{\gamma_{12}\gamma_{21}}) = 0, \tag{27}$$

where s_1^* and s_2^* are $O(1)$ constants defined by $s_1^* = s_1\epsilon$, $s_2^* = s_2\epsilon$. We note that this leading order bifurcation condition has the same structure as Steigmann and Ogden's (1997) Eq. (6.30), but they are not equivalent (see the discussion in section 4). In the next section we shall derive, from the original governing Eqs. (1) and (5a), the asymptotic governing equations that yield the same leading order bifurcation condition (27).

We now suppose that the coating is even stiffer. The next case of interest is clearly when

$$\alpha = O(1/\epsilon^2). \tag{28}$$

When this relation holds, an inspection of (25) shows that the third term is the only dominant term. By equating this term to zero we find that $\lambda = 1$, which means that to leading order there must be no pre-strain. The next biggest term in (25) is the second term which is $O(1/\epsilon)$. In order for the third term to balance this term, we must have $\lambda - 1 = O(\epsilon)$. Thus we write

$$\lambda = 1 + \epsilon\phi, \tag{29}$$

where ϕ is an $O(1)$ constant. We recall that all the moduli $\mathscr{A}_{jilk}$ and $\mathscr{A}_{jilk}$ are functions of the stretch λ. We may expand these moduli about $\lambda = 1$ to obtain

$$\bar{\mathscr{A}}_{jilk} = \bar{\mathscr{A}}_{jilk}^0 + \epsilon\phi\bar{\mathscr{A}}_{jilk}' + \cdots, \quad \mathscr{A}_{jilk} = \mathscr{A}_{jilk}^0 + \epsilon\phi\mathscr{A}_{jilk}' + \cdots, \tag{30}$$

where

$$\bar{\mathscr{A}}_{jilk}^0 = \bar{\mathscr{A}}_{jilk}\,|_{\lambda=1} = \bar{\lambda}^*\delta_{ji}\delta_{lk} + \bar{\mu}(\delta_{jl}\delta_{ik} + \delta_{jk}\delta_{il}),$$

$$\mathscr{A}^0_{jilk} = \mathscr{A}_{jilk}\mid_{\lambda=1} = \lambda^*\delta_{ji}\delta_{lk} + \mu(\delta_{jl}\delta_{ik} + \delta_{jk}\delta_{il}),$$

$$\bar{\mathscr{A}}'_{jilk} = \frac{\partial\bar{\mathscr{A}}_{jilk}}{\partial\lambda}\mid_{\lambda=1}, \quad \mathscr{A}'_{jilk} = \frac{\partial\mathscr{A}_{jilk}}{\partial\lambda}\mid_{\lambda=1}.$$

In the above expressions, $(\bar{\lambda}^*, \bar{\mu})$ and (λ^*, μ) are the Lamé constants for the coating and half-space, respectively (we have added a '*' to λ to avoid confusion with the principal stretch). We may also write down the corresponding expansions for $\bar{\alpha}_{ij}$, α_{ij}, etc but it suffices to remind the reader of the relations (4). On substituting (29) and (30) into (20) and equating the coefficients of $1/\epsilon$, we obtain

$$\phi = \frac{2\mu(\lambda^* + 2\mu)}{(\bar{\gamma}'_{21} - \bar{\gamma}'_{12})(\lambda^* + 3\mu)} = \frac{2\mu(\lambda^* + 2\mu)}{(\bar{\mathscr{A}}'_{1221} - \bar{\mathscr{A}}'_{1212})(\lambda^* + 3\mu)},$$

where we have scaled $\bar{\gamma}_{21}$, $\bar{\gamma}_{12}$, $\mathscr{A}_{1221}$, $\mathscr{A}_{1212}$ by $1/\epsilon^2$. Hence the bifurcation condition is given by

$$\lambda = 1 - \frac{2\mu(\lambda^* + 2\mu)}{\bar{\sigma}_{1\lambda}(\lambda^* + 3\mu)}\epsilon + O(\epsilon^2), \tag{31}$$

where

$$\bar{\sigma}_{1\lambda} = \left.\frac{\partial\bar{\sigma}_1}{\partial\lambda}\right|_{\lambda=1} = \bar{\mathscr{A}}'_{1212} - \bar{\mathscr{A}}'_{1221},$$

obtained by making use of (3).

It can easily be shown that in the absence of half-space, the bifurcation condition for the layer expands like $\lambda = 1 + O(\epsilon^2)$ in the limit $\epsilon \to 0$. Thus the second term on the right hand side of (31) is due to the existence of the half-space (this is also confirmed by the fact that if the moduli λ^* and μ for the half-space were much smaller than we have assumed, the second term on the right hand side of (31) would be much smaller than $O(\epsilon)$).

Finally, we consider even stronger coating described by $\alpha = O(1/\epsilon^3)$. We shall show in section 3 that this case corresponds to the classical model for beams on elastic foundations.

An inspection of (25) shows that in this case a proper balance can be achieved if $\lambda = 1 + O(\epsilon^2)$. Then the third term becomes of order $1/\epsilon^2$ and it is balanced by the second and fifth terms in (25). Thus we write

$$\lambda = 1 + \epsilon^2\psi, \tag{32}$$

where ψ is an $O(1)$ constant, and expand the moduli $\mathscr{A}_{jilk}$ and $\mathscr{A}_{jilk}$ as

$$\bar{\mathscr{A}}_{jilk} = \bar{\mathscr{A}}^0_{jilk} + \epsilon^2\psi\bar{\mathscr{A}}'_{jilk} + \cdots, \quad \mathscr{A}_{jilk} = \mathscr{A}^0_{jilk} + \epsilon^2\psi\mathscr{A}'_{jilk} + \cdots. \tag{33}$$

On substituting (32) and (33) into (20) and equating the coefficients of $1/\epsilon^2$, we obtain

$$\psi = -\frac{2\mu(\lambda^* + 2\mu)}{\bar{\sigma}_{1\lambda}(\lambda^* + 3\mu)} - \frac{\bar{\mu}(\bar{\lambda}^* + \bar{\mu})}{3\bar{\sigma}_{1\lambda}(\bar{\lambda}^* + 2\bar{\mu})}, \tag{34}$$

where we have scaled $\bar{\lambda}^*$, $\bar{\mu}$, $\bar{\sigma}_1$ by $1/\epsilon^3$. Hence the bifurcation condition is given by

$$\lambda = 1 - \left\{ \frac{2\mu(\lambda^* + 2\mu)}{\bar{\sigma}_{1\lambda}(\lambda^* + 3\mu)} + \frac{\bar{\mu}(\bar{\lambda}^* + \bar{\mu})}{3\bar{\sigma}_{1\lambda}(\bar{\lambda}^* + 2\bar{\mu})} \right\} \epsilon^2 + O(\epsilon^4). \tag{35}$$

We note that the bifurcation condition (31) in the previous case can actually be recovered from (35) by assuming that $\bar{\mu}$ and $\bar{\lambda}^*$ are of order ϵ (note that the moduli of the coating for this case and the previous case have been scaled by $1/\epsilon^2$ and $1/\epsilon^3$, respectively).

3. Asymptotic theory

The principal objective of an asymptotic theory for a coated elastic half-space is to simplify analysis by incorporating all the effects of the coating into the boundary conditions (i.e. the interfacial conditions) for the half-space. In this section we derive the leading order asymptotic interfacial conditions which yield the bifurcation conditions derived in the previous section. Since as we just observed the case $\mathscr{A}_{jilk} = O(1/\epsilon^2)$ can be recovered from the case $\mathscr{A}_{jilk} = O(1/\epsilon^3)$ by taking the appropriate limit, we shall only consider two cases: $\mathscr{A}_{jilk} = O(1/\epsilon)$ and $\mathscr{A}_{jilk} = O(1/\epsilon^3)$. The first limit corresponds to the case when the half-space and coating exerts maximum effects on each other and buckling takes place at finite strains, whereas the second limit corresponds to the case that deflections are large enough for bending effects to become important.

3.1. Case I: $\mathscr{A}_{jilk} = O(1/\epsilon)$

In this case we scale $\mathscr{A}_{jilk}$ by $1/\epsilon$ and use the same symbols to denote the scaled quantities. We also scale $(\bar{u}_1, \bar{u}_2)$ and (x_1, x_2) through

$$(\bar{u}_1, \bar{u}_2) = (u/k, v/k), \quad (x_1, x_2) = (x/k, hy),$$

where $1/k$ is now viewed as a reference lengthscale. The original governing Eq. (5a) and the expression (5b) for the traction then become

$$\bar{\mathscr{A}}_{2121} u_{yy} + (\bar{\mathscr{A}}_{1122} + \bar{\mathscr{A}}_{2112}) \epsilon v_{xy} + \bar{\mathscr{A}}_{1111} \epsilon^2 u_{xx} = 0, \tag{36}$$

$$\bar{\mathscr{A}}_{2222} v_{yy} + (\bar{\mathscr{A}}_{1221} + \bar{\mathscr{A}}_{2211}) \epsilon u_{xy} + \bar{\mathscr{A}}_{1212} \epsilon^2 v_{xx} = 0, \tag{37}$$

$$\bar{T}_1 = \frac{1}{\epsilon^2} (\bar{\mathscr{A}}_{2121} u_y + \epsilon \bar{\mathscr{A}}_{2112} v_x), \quad \bar{T}_2 = \frac{1}{\epsilon^2} (\bar{\mathscr{A}}_{2222} v_y + \epsilon \bar{\mathscr{A}}_{2211} u_x), \tag{38}$$

where suffices x and y signify differentiation. We look for an asymptotic solution of the form

$$u = u^{(0)} + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \cdots, \quad v = v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \cdots.$$

On substituting these expressions into (36)–(38) and equating the coefficients of like powers of ϵ, we obtain the following hierarchy of differential equations:

$O(1)$:

$$u^{(0)}_{yy} = 0, \quad v^{(0)}_{yy} = 0, \tag{39}$$

$O(\epsilon)$:

$$\bar{\mathscr{A}}_{2121}u_{yy}^{(1)} = -(\bar{\mathscr{A}}_{1122} + \bar{\mathscr{A}}_{2112})v_{xy}^{(0)}, \quad \bar{\mathscr{A}}_{2222}v_{yy}^{(1)} = -(\bar{\mathscr{A}}_{1221} + \bar{\mathscr{A}}_{2211})u_{xy}^{(0)}, \tag{40}$$

$O(\epsilon^2)$:

$$\bar{\mathscr{A}}_{2121}u_{yy}^{(2)} = -(\bar{\mathscr{A}}_{1122} + \bar{\mathscr{A}}_{2112})v_{xy}^{(1)} - \bar{\mathscr{A}}_{1111}u_{xx}^{(0)}, \tag{41}$$

$$\bar{\mathscr{A}}_{2222}v_{yy}^{(2)} = -(\bar{\mathscr{A}}_{1221} + \bar{\mathscr{A}}_{2211})u_{xy}^{(1)} - \bar{\mathscr{A}}_{1212}v_{xx}^{(0)}. \tag{42}$$

The corresponding boundary conditions which must hold on $y = 0, 1$ are

$O(1)$:

$$u_y^{(0)} = v_y^{(0)} = 0, \tag{43}$$

$O(\epsilon)$:

$$\bar{\mathscr{A}}_{2121}u_y^{(1)} = -\bar{\mathscr{A}}_{2112}v_x^{(0)}, \quad \bar{\mathscr{A}}_{2222}v_y^{(1)} = -\bar{\mathscr{A}}_{2211}u_x^{(0)}, \tag{44}$$

$O(\epsilon^2)$:

$$\bar{T}_1 = \bar{\mathscr{A}}_{2121}u_y^{(2)} + \bar{\mathscr{A}}_{2112}v_x^{(1)}, \quad \bar{T}_2 = \bar{\mathscr{A}}_{2222}v_y^{(2)} + \bar{\mathscr{A}}_{2211}u_x^{(1)}, \tag{45}$$

where we have anticipated that the traction at the interface is of $O(1)$, as required by the scalings for the half-space.

Eqs. (39) and the boundary conditions (43) imply that

$$u^{(0)} = u^{(0)}(x), \quad v^{(0)} = v^{(0)}(x). \tag{46}$$

Solving the second order problem (40) and (44) yields

$$u_y^{(1)} = -\frac{\bar{\mathscr{A}}_{2112}}{\bar{\mathscr{A}}_{2121}}v_x^{(0)}, \quad v_y^{(1)} = -\frac{\bar{\mathscr{A}}_{2211}}{\bar{\mathscr{A}}_{2222}}u_x^{(0)}. \tag{47}$$

In order to derive the governing equations for $u^{(0)}(x)$ and $v^{(0)}(x)$, we make use of (47) and re-write (41) and (42) as

$$\frac{\partial}{\partial y}(\bar{\mathscr{A}}_{2121}u_y^{(2)} + \bar{\mathscr{A}}_{2112}v_x^{(1)}) = \left(\frac{\bar{\mathscr{A}}_{1122}\bar{\mathscr{A}}_{1122}}{\bar{\mathscr{A}}_{2222}} - \bar{\mathscr{A}}_{1111}\right)u_{xx}^{(0)},$$

$$\frac{\partial}{\partial y}(\bar{\mathscr{A}}_{2222}v_y^{(2)} + \bar{\mathscr{A}}_{2211}u_x^{(1)}) = \left(\frac{\bar{\mathscr{A}}_{1221}^2}{\bar{\mathscr{A}}_{2121}} - \bar{\mathscr{A}}_{1212}\right)v_{xx}^{(0)}.$$

On solving these equations subject to the boundary conditions (45), we obtain

$$\left(\frac{\bar{\mathscr{A}}_{1122}^2}{\bar{\mathscr{A}}_{2222}} - \bar{\mathscr{A}}_{1111}\right)u_{xx}^{(0)} = \bar{T}_1(1) - \bar{T}_1(0), \quad \left(\frac{\bar{\mathscr{A}}_{1221}^2}{\bar{\mathscr{A}}_{2121}} - \bar{\mathscr{A}}_{1212}\right)v_{xx}^{(0)} = \bar{T}_2(1) - \bar{T}_2(0). \tag{48}$$

Since both displacement components are continuous at $y = 0$, the $u^{(0)}$, $v^{(0)}$ in the above expression can be replaced by the corresponding components for the half-space evaluated at $x_2 = y = 0$. Thus all the effects of the coating are embodied in these two interfacial conditions.

The governing Eq. (1) applies to the half-space. We scale u_1, u_2, x_1, x_2 by $1/k$ and use the same symbols to denote the scaled quantities. The displacement field in the half-space then satisfies

$$\mathscr{A}_{1111}u_{1,11} + (\mathscr{A}_{1122} + \mathscr{A}_{2112})u_{2,12} + \mathscr{A}_{2121}u_{1,22} = 0, \tag{49}$$

$$\mathscr{A}_{1212}u_{2,11} + (\mathscr{A}_{1221} + \mathscr{A}_{2211})u_{1,12} + \mathscr{A}_{2222}u_{2,22} = 0, \tag{50}$$

The continuity conditions at the interface are

$$u_1(x_1, 0) = u^{(0)}(x_1), \quad u_2(x_1, 0) = v^{(0)}(x_1), \tag{51}$$

$$\mathscr{A}_{2112}u_{2,1} + \mathscr{A}_{2121}u_{1,2} = \bar{T}_1(0), \quad \mathscr{A}_{2211}u_{1,1} + \mathscr{A}_{2222}u_{2,2} = \bar{T}_2(0). \tag{52}$$

We assume that the top surface at $y = 1$ is traction free so that $\bar{T}_1(1) = \bar{T}_2(1) = 0$. On eliminating $\bar{T}_1(0)$ and $\bar{T}_2(0)$ from (48) and (52) and making use of (51), we obtain

$$\mathscr{A}_{2112}u_{2,1} + \mathscr{A}_{2121}u_{1,2} = \left(\bar{\mathscr{A}}_{1111} - \frac{\bar{\mathscr{A}}_{1122}^2}{\bar{\mathscr{A}}_{2222}}\right)u_{1,11}, \quad \text{on} \quad x_2 = 0, \tag{53}$$

$$\mathscr{A}_{2211}u_{1,1} + \mathscr{A}_{2222}u_{2,2} = \left(\bar{\mathscr{A}}_{1212} - \frac{\bar{\mathscr{A}}_{1221}^2}{\bar{\mathscr{A}}_{2121}}\right)u_{2,11}, \quad \text{on} \quad x_2 = 0. \tag{54}$$

Thus the original problem is reduced, to leading order, to solving (49) and (50) subject to (53), (54) and the decay conditions u_1, $u_2 \to 0$ as $x_2 \to -\infty$.

We look for a buckling solution of the form

$$u_1 = H_1(x_2)e^{ix_1}, \quad u_2 = H_2(x_2)e^{ix_1}, \tag{55}$$

where the non-dimensional mode number is unity because we have chosen the inverse of the dimensional mode number k to be our reference lengthscale. Following the standard procedure of linear stability analysis, it can easily be shown that H_1 and H_2 have a non-trivial solution only if the bifurcation condition (27) is satisfied.

3.2. Case II: $\mathscr{A}_{jilk} = O(1/\epsilon^3)$

In this case, the principal stretch expands as in (32). Guided by the classical plate and beam theories, we assume for the layer that $\bar{u}_1 = O(\epsilon\bar{u}_2)$ and write

$$(\bar{u}_1, \bar{u}_2) = (hu, v/k), \quad (x_1, x_2) = (x/k, hy).$$

We scale $\mathscr{A}_{jilk}$ by $1/\epsilon^3$ and use the same symbols for the scaled moduli.

With the scaled moduli expanded as in (33), the governing Eqs. (5a) for the coating become

$$\bar{\mathscr{A}}^0_{2121}u_{yy} + (\bar{\mathscr{A}}^0_{1122} + \bar{\mathscr{A}}^0_{2112})v_{xy} + \epsilon^2\psi\bar{\mathscr{A}}'_{2121}u_{yy} + \epsilon^2\psi(\bar{\mathscr{A}}'_{1122} + \bar{\mathscr{A}}'_{2112})v_{xy} + \frac{1}{2}\epsilon^4\psi^2\bar{\mathscr{A}}''_{2121}u_{yy} + \frac{1}{2}\epsilon^4\psi^2(\bar{\mathscr{A}}''_{1122} + \bar{\mathscr{A}}''_{2112})v_{xy} + \epsilon^2\bar{\mathscr{A}}^0_{1111}u_{xx} + \epsilon^4\psi\bar{\mathscr{A}}'_{1111}u_{xx} = O(\epsilon^5), \tag{56}$$

$$\bar{\mathscr{A}}^0_{2222}v_{yy} + \epsilon^2\psi\bar{\mathscr{A}}'_{2222}v_{yy} + \epsilon^2[(\bar{\mathscr{A}}^0_{1221} + \bar{\mathscr{A}}^0_{2211})u_{xy} + \bar{\mathscr{A}}^0_{1212}v_{xx}] + \frac{1}{2}\epsilon^4\psi^2\bar{\mathscr{A}}''_{2222}v_{yy} + \epsilon^4\psi[(\bar{\mathscr{A}}'_{1221} + \bar{\mathscr{A}}'_{2211})u_{xy} + \bar{\mathscr{A}}'_{1212}v_{xx}] = O(\epsilon^5), \tag{57}$$

whilst the traction components from (5b) have the expressions

$$\epsilon^3\bar{T}_1 = (\bar{\mathscr{A}}^0_{2112} + \epsilon^2\psi\bar{\mathscr{A}}'_{2112} + \frac{1}{2}\epsilon^4\psi^2\bar{\mathscr{A}}'_{2112})(v_x + u_y) + O(\epsilon^5), \tag{58}$$

$$\epsilon^4\bar{T}_2 = \bar{\mathscr{A}}^0_{2222}v_y + \epsilon^2\psi\bar{\mathscr{A}}'_{2222}v_y + \epsilon^4\psi^2\bar{\mathscr{A}}''_{2222}v_y + \epsilon^2\bar{\mathscr{A}}^0_{2211}u_x + \epsilon^4\psi\bar{\mathscr{A}}'_{2211}u_x + O(\epsilon^5). \tag{59}$$

We look for a solution of the form

$$u = u^{(0)} + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \epsilon^3 u^{(3)} + \epsilon^4 u^{(4)} + \cdots,$$

$$v = v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \epsilon^3 v^{(3)} + \epsilon^4 v^{(4)} + \cdots.$$

On substituting these expressions into (56) and (57) and equating coefficients of like powers of ϵ, we obtain the following hierarchy of differential equations:

$O(1)$:

$$\bar{\mathscr{A}}^0_{2121}u^{(0)}_{yy} + (\bar{\mathscr{A}}^0_{1122} + \bar{\mathscr{A}}^0_{2112})v^{(0)}_{xy} = 0, \tag{60}$$

$$\bar{\mathscr{A}}^0_{2222}v^{(0)}_{yy} = 0, \tag{61}$$

$O(\epsilon)$:

$$\bar{\mathscr{A}}^0_{2121}u^{(1)}_{yy} + (\bar{\mathscr{A}}^0_{1122} + \bar{\mathscr{A}}^0_{2112})v^{(1)}_{xy} = 0, \tag{62}$$

$$\bar{\mathscr{A}}^0_{2222}v^{(1)}_{yy} = 0, \tag{63}$$

$O(\epsilon^2)$:

$$\bar{\mathscr{A}}^0_{2121}u^{(2)}_{yy} + (\bar{\mathscr{A}}^0_{1122} + \bar{\mathscr{A}}^0_{2112})v^{(2)}_{xy} = -\psi\bar{\mathscr{A}}'_{2121}u^{(0)}_{yy} - \psi(\bar{\mathscr{A}}'_{1122} + \bar{\mathscr{A}}'_{2112})v^{(0)}_{xy} - \bar{\mathscr{A}}^0_{1111}u^{(0)}_{xx}, \tag{64}$$

$$\bar{\mathscr{A}}^0_{2222}v^{(2)}_{yy} = -\psi\bar{\mathscr{A}}'_{2222}v^{(0)}_{yy} - (\bar{\mathscr{A}}^0_{1221} + \bar{\mathscr{A}}^0_{2211})u^{(0)}_{xy} - \bar{\mathscr{A}}^0_{1212}v^{(0)}_{xx}, \tag{65}$$

$O(\epsilon^3)$:

$$\bar{\mathscr{A}}^0_{2121}u^{(3)}_{yy} + (\bar{\mathscr{A}}^0_{1122} + \bar{\mathscr{A}}^0_{2112})v^{(3)}_{xy} = -\psi\bar{\mathscr{A}}'_{2121}u^{(1)}_{yy} - \psi(\bar{\mathscr{A}}'_{1122} + \bar{\mathscr{A}}'_{2112})v^{(1)}_{xy} - \bar{\mathscr{A}}^0_{1111}u^{(1)}_{xx}, \tag{66}$$

$$\mathscr{A}^0_{2222}v^{(3)}_{yy} = -\psi\bar{\mathscr{A}}'_{2222}v^{(1)}_{yy} - (\bar{\mathscr{A}}^0_{1221} + \bar{\mathscr{A}}^0_{2211})u^{(1)}_{xy} - \bar{\mathscr{A}}^0_{1212}v^{(1)}_{xx}, \tag{67}$$

$O(\epsilon^4)$:

$$\begin{aligned}\bar{\mathscr{A}}^0_{2121}u^{(4)}_{yy} &+ (\bar{\mathscr{A}}^0_{1122} + \bar{\mathscr{A}}^0_{2112})v^{(4)}_{xy} \\ &= -\psi\bar{\mathscr{A}}'_{2121}u^{(2)}_{yy} - \psi(\bar{\mathscr{A}}'_{1122} + \bar{\mathscr{A}}'_{2112})v^{(2)}_{xy} - \tfrac{1}{2}\psi^2\bar{\mathscr{A}}''_{2121}u^{(0)}_{yy} - \tfrac{1}{2}\psi^2(\bar{\mathscr{A}}''_{1122} + \bar{\mathscr{A}}''_{2112})v^{(0)}_{xy} \\ &\quad - \bar{\mathscr{A}}^0_{1111}u^{(2)}_{xx} - \psi\bar{\mathscr{A}}'_{1111}u^{(0)}_{xx},\end{aligned} \tag{68}$$

$$\begin{aligned}\bar{\mathscr{A}}^0_{2222}v^{(4)}_{yy} = -\psi\bar{\mathscr{A}}'_{2222}v^{(2)}_{yy} - \tfrac{1}{2}\psi^2\bar{\mathscr{A}}''_{2222}v^{(0)}_{yy} - \bar{\mathscr{A}}^0_{1212}v^{(2)}_{xx} - (\bar{\mathscr{A}}^0_{1221} + \bar{\mathscr{A}}^0_{2211})u^{(2)}_{xy} - \psi(\bar{\mathscr{A}}'_{1221} \\ + \bar{\mathscr{A}}'_{2211})u^{(0)}_{xy} - \psi\bar{\mathscr{A}}'_{1212}v^{(0)}_{xx}.\end{aligned} \tag{69}$$

The corresponding boundary conditions which must hold on $y = 0$, 1 are obtained from (58) and (59) and are given by

$O(1)$:

$$\bar{\mathscr{A}}^0_{2112}(v^{(0)}_x + u^{(0)}_y) = 0, \quad \bar{\mathscr{A}}^0_{2222}v^{(0)}_y = 0, \tag{70}$$

$O(\epsilon)$:

$$\bar{\mathscr{A}}^0_{2112}(v^{(1)}_x + u^{(1)}_y) = 0, \quad \bar{\mathscr{A}}^0_{2222}v^{(1)}_y = 0, \tag{71}$$

$O(\epsilon^2)$:

$$\epsilon T_1 = \bar{\mathscr{A}}^0_{2112}(v^{(2)}_x + u^{(2)}_y) + \psi\bar{\mathscr{A}}'_{2112}(v^{(0)}_x + u^{(0)}_y), \tag{72}$$

$$\bar{\mathscr{A}}^0_{2222}v^{(2)}_y = -\psi\bar{\mathscr{A}}'_{2222}v^{(0)}_y - \bar{\mathscr{A}}^0_{2211}u^{(0)}_x, \tag{73}$$

$O(\epsilon^3)$:

$$T_1 = \bar{\mathscr{A}}^0_{2112}(v^{(3)}_x + u^{(3)}_y) + \psi\bar{\mathscr{A}}'_{2112}(v^{(1)}_x + u^{(1)}_y), \tag{74}$$

$$\bar{\mathscr{A}}^0_{2222}v^{(3)}_y = -\psi\bar{\mathscr{A}}'_{2222}v^{(1)}_y - \bar{\mathscr{A}}^0_{2211}u^{(1)}_x, \tag{75}$$

$O(\epsilon^4)$:

$$T_2 = \bar{\mathscr{A}}^0_{2222}v^{(4)}_y + \psi\bar{\mathscr{A}}'_{2222}v^{(2)}_y + \psi^2\bar{\mathscr{A}}''_{2222}v^{(0)}_y + \bar{\mathscr{A}}^0_{2211}u^{(2)}_x + \psi\bar{\mathscr{A}}'_{2211}u^{(0)}_x. \tag{76}$$

In writing down these boundary conditions we have assumed that $T_2 = O(1)$ and ϵT_1 is $O(1)$ or smaller. We shall eventually show that satisfaction of the interfacial traction continuity conditions requires $T_1 = O(1)$. The reason that we allow the possibility $\epsilon T_1 = O(1)$ is that we want to show that it is this (wrong) scaling that yields the interfacial conditions used by Shield et al. (1994). The rule followed in the following analysis is that if ϵT_1 were indeed $O(1)$, then the leading order analysis should stop at (72) and condition (74) need not be considered, whereas if ϵT_1 is of order ϵ, i.e. $T_1 = O(1)$, we simply set $\epsilon T_1 = 0$ on the left hand of (72) and deduce one of the interfacial traction continuity conditions from (74).

Solving (60), (61) subject to (70), and (62), (63) subject to (71), we obtain

$$v^{(0)} = W_2(x), \quad u^{(0)} = W_1(x) - yW_2'(x), \tag{77}$$

$$v^{(1)} = Z_2(x), \quad u^{(1)} = Z_1(x) - yZ_2'(x), \tag{78}$$

where W_1, W_2, Z_1 and Z_2 are arbitrary functions of x. Solving (65) subject to (73), (67) subject to (75), we obtain

$$v^{(2)} = -\frac{\bar{\lambda}^*}{\bar{\lambda}^* + 2\bar{\mu}}\left(yW_1' - \frac{1}{2}y^2W_2''\right) + f_3(x), \tag{79}$$

$$v^{(3)} = -\frac{\bar{\lambda}^*}{\bar{\lambda}^* + 2\bar{\mu}}\left(yZ_1' - \frac{1}{2}y^2Z_2''\right) + g_3(x), \tag{80}$$

where f_3 and g_3 are arbitrary functions of x. With $v^{(2)}$ and $v^{(3)}$ known, Eqs. (64) and (66) can be solved for $u^{(2)}$ and $u^{(3)}$, respectively. We have

$$u_y^{(2)} = -\frac{3\bar{\lambda}^* + 4\bar{\mu}}{\bar{\lambda}^* + 2\bar{\mu}}\left(yW_1'' - \frac{1}{2}y^2W_2'''\right) - \frac{\bar{\lambda}^* + \bar{\mu}}{\bar{\mu}}f_3' + \frac{1}{\bar{\mu}}f_2(x), \tag{81}$$

$$u_y^{(3)} = -\frac{3\bar{\lambda}^* + 4\bar{\mu}}{\bar{\lambda}^* + 2\bar{\mu}}\left(yZ_1'' - \frac{1}{2}y^2Z_2'''\right) - \frac{\bar{\lambda}^* + \bar{\mu}}{\bar{\mu}}g_3' + \frac{1}{\bar{\mu}}g_2(x), \tag{82}$$

where f_2 and g_2 are another two functions arising from integrations. On substituting these solutions into the boundary conditions (72) and (74), we obtain

$$\frac{4\bar{\mu}(\bar{\lambda}^* + \bar{\mu})}{\bar{\lambda}^* + 2\bar{\mu}}\left(W_1'' - \frac{1}{2}W_2'''\right) = f_2 - \bar{\lambda}^* f_3' = T_1\epsilon, \tag{83}$$

$$\frac{4\bar{\mu}(\bar{\lambda}^* + \bar{\mu})}{\bar{\lambda}^* + 2\bar{\mu}}\left(Z_1'' - \frac{1}{2}Z_2'''\right) = g_2 - \bar{\lambda}^* g_3' = T_1. \tag{84}$$

Finally, substituting the above solutions into (69) and (76) and re-arranging, we obtain

$$\frac{\partial}{\partial y}\{(\bar{\lambda}^* + 2\bar{\mu})v_y^{(4)} + \bar{\lambda}^* u_x^{(2)}\}$$
$$= \psi\left(\bar{\mathscr{A}}'_{1221} + \bar{\mathscr{A}}'_{2211} - \bar{\mathscr{A}}'_{1212} - \frac{\bar{\lambda}^*}{\bar{\lambda}^* + 2\bar{\mu}}\bar{\mathscr{A}}'_{2222}\right)W_2'' + \frac{4\bar{\mu}(\bar{\lambda}^* + \bar{\mu})}{\bar{\lambda}^* + 2\bar{\mu}}\left\{(y-1)W_1''' - \frac{1}{2}(y^2 - 1)W_2''''\right\}, \tag{85}$$

and

$$(\bar{\lambda}^* + 2\bar{\mu})v_y^{(4)} + \bar{\lambda}^* u_x^{(2)} = T_2 + \psi\left(\frac{\bar{\lambda}^*}{\bar{\lambda}^* + 2\bar{\mu}}\bar{\mathscr{A}}'_{2222} - \bar{\mathscr{A}}'_{2211}\right)(W_1' - yW_2''). \tag{86}$$

On integrating (85) from $y = 0$ to $y = 1$ and making use of the boundary conditions (86), we obtain

$$\frac{4\bar{\mu}(\bar{\lambda}^* + \bar{\mu})}{\bar{\lambda}^* + 2\bar{\mu}}\left(\frac{1}{2}W_1''' - \frac{1}{3}W_2''''\right) - \psi(\bar{\mathscr{A}}'_{1221} - \bar{\mathscr{A}}'_{1212})W_2'' = T_2(0). \tag{87}$$

If we express the Lamé constants $\bar{\lambda}^*$ and $\bar{\mu}$ in terms of the Young's modulus and Poisson's ratio for the coating, then the left hand sides of (83) and (87), written in terms of the unscaled variables, are identical to the corresponding terms in Shield et al.'s (1994) Eq. (12). The left hand sides of (83) and (87) are similar, but not identical, to those of Steigmann and Ogden's (1997) Eq. (6.19).

It remains to match, at the interface, the solutions we have just found for the coating with those for the half-space. The conditions at the interface are given by (51) and (52). Since we have assumed for the coating that $\bar{u}_1 = O(\epsilon)$, $\bar{u}_2 = O(1)$, it may seem at first sight that we should have $u_1 = O(\epsilon)$, $u_2 = O(1)$ for the half-space. However, if these order relations hold, then (49) gives $u_{2,12} = 0$ to leading order, which clearly does not admit a non-trivial solution which is periodic in the x_1-direction. Thus we must have $u_1 = O(1)$, $u_2 = O(1)$ for the half-space and we can impose the condition $u_1(x_1, 0) = O(\epsilon)$ in order to satisfy the matching requirement $u_1(x_1, 0) = \bar{u}_1(x_1, 0)$. It then follows from (52) that $T_1(0) = O(1)$ and $T_2(0) = O(1)$. The result $T_2(0) = O(1)$ is consistent with (87). The right hand side of (83) now becomes $O(\epsilon)$ and must be set to zero (following the rule stated in the paragraph below (76)). Hence $W_1'' = W_2'''/2$ and Eq. (87) reduces to

$$-\frac{\bar{\mu}(\bar{\lambda}^* + \bar{\mu})}{3(\bar{\lambda}^* + 2\bar{\mu})}W_2'''' - \psi(\bar{\mathscr{A}}'_{1221} - \bar{\mathscr{A}}'_{1212})W_2'' = T_2(0), \tag{88}$$

which resembles Steigmann and Ogden's (1997) Eq. (6.19a). Eq. (84) now becomes

$$\frac{4\bar{\mu}(\bar{\lambda}^* + \bar{\mu})}{\bar{\lambda}^* + 2\bar{\mu}}\left(Z_1'' - \frac{1}{2}Z_2'''\right) = T_1, \tag{89}$$

which, as we shall see shortly, is not involved in the leading order analysis. We note that the displacement in the coating expands as

$$\bar{u}_1 = h(W_1 - yW_2') + h^2(Z_1 - yZ_2') + O(h^3), \quad \bar{u}_2 = W_2 + O(h).$$

For the half-space we again scale u_1, u_2, x_1, x_2 for the half-space by $1/k$ and use the same symbols to denote the scaled quantities. Then to leading order the governing equations for the half-space are

$$(\lambda^* + 2\mu)u_{1,11} + (\lambda^* + \mu)u_{2,12} + \mu u_{1,22} = 0, \tag{90}$$

$$\mu u_{2,11} + (\lambda^* + \mu)u_{1,12} + (\lambda^* + 2\mu)u_{2,22} = 0. \tag{91}$$

These are to be solved subjected to the leading order interfacial conditions

$$u_1(x_1, 0) = 0, \quad u_2(x_1, 0) = v^{(0)}(x_1) = W_2(x_1), \tag{92}$$

$$\mu(u_{2,1} + u_{1,2}) = \frac{4\bar{\mu}(\bar{\lambda}^* + \bar{\mu})}{\bar{\lambda}^* + 2\bar{\mu}}\left(Z_1'' - \frac{1}{2}Z_2'''\right), \tag{93}$$

$$\lambda^* u_{1,1} + (\lambda^* + 2\mu)u_{2,2} = -\frac{\bar{\mu}(\bar{\lambda}^* + \bar{\mu})}{3(\bar{\lambda}^* + 2\bar{\mu})}W_2'''' - \psi(\bar{\mathscr{A}}'_{1221} - \bar{\mathscr{A}}'_{1212})W_2''. \tag{94}$$

With the use of (92a), Eq. (94) reduces to

$$(\lambda^* + 2\mu)u_{2,2} = -\frac{\bar{\mu}(\bar{\lambda}^* + \bar{\mu})}{3(\bar{\lambda}^* + 2\bar{\mu})}W_2'''' - \psi(\bar{\mathscr{A}}'_{1221} - \bar{\mathscr{A}}'_{1212})W_2''. \tag{95}$$

The original problem is now reduced to solving (90) and (91) subject to (92a, b) and (95). We look for a buckling solution of the form (55). On substituting (55) into (90) and (91) and solving the resulting ordinary differential equations, we obtain

$$H_1(x_2) = (A + Bx_2)e^{x_2}, \quad iH_2(x_2) = \left(A - \frac{\lambda^* + 3\mu}{\lambda^* + \mu}B + Bx_2\right)e^{x_2}. \tag{96}$$

The boundary condition (92b) yields

$$W_2(x_1) = H_2(0)e^{ix_1}, \tag{97}$$

and after canceling the common factor e^{ix_1} the boundary condition (95) becomes

$$(\lambda^* + 2\mu)H_2'(0) = -\frac{\bar{\mu}(\bar{\lambda}^* + \bar{\mu})}{3(\bar{\lambda}^* + 2\bar{\mu})}H_2(0) + \psi(\bar{\mathscr{A}}'_{1221} - \bar{\mathscr{A}}'_{1212})H_2(0). \tag{98}$$

The boundary condition (92a) implies $H_1(0) = 0$ and so from (96a) we have $A = 0$. On substituting (96b) into (98), we obtain the same expression for ψ as in (34), and hence the same bifurcation condition as in section 2.

4. Discussion

In this paper we have examined the asymptotic properties of the exact bifurcation condition for the structure of a thin layer bonded to a half-space. Two asymptotic limits are identified for small kh: $\mathscr{A}_{jilk} = O(kh\bar{\mathscr{A}}_{jilk})$ and $\mathscr{A}_{jilk} = O(k^3h^3\bar{\mathscr{A}}_{jilk})$. For each limit we carried out an asymptotic analysis based on the original governing equations. The analysis reduces the original problem to a simplified problem in which all effects of the coating are incorporated into the boundary conditions for the half-space. We showed that the reduced problem yielded the same bifurcation condition as that derived from the exact bifurcation condition.

In Shield et al.'s (1994) analysis, the strains are assumed to be small so that all the elastic moduli are evaluated at zero strains (i.e. at $\lambda = 1$). Our analysis shows that this is justified if $\mathscr{A}_{jilk} = O(k^3h^3\mathscr{A}_{jilk})$. For this case our leading order traction continuity condition (87) is identical to their Eq. (12a), but their (12b) is inconsistent in the sense that in this equation $u_{xx} - hv_{xx}/2$ and T_{yx} are of different orders of magnitude. The correct form of the shear traction continuity condition is given by our Eq. (89) which does not appear in the leading order analysis. The latter fact implies that when the coating is much stiffer than the half-space, the shearing at the interface can be neglected and the coating behaves as if it were supported by an array of springs. In the engineering theory for plates on elastic foundations, the elastic foundations are usually modeled by an array of springs. Our present analysis provides a mathematical justification for this assumption.

Steigmann and Ogden's (1997) theory assumes that bifurcation takes place at finite strains. Our analysis shows that this is only possible if the half-space is stiffer than in the previous case so that it has a stronger influence. More precisely, we need $\mathscr{A}_{jilk} = O(kh\mathscr{A}_{jilk})$ or $\mathscr{A}_{jilk} \gg O(kh\mathscr{A}_{jilk})$. However in the latter case the coating has a negligible effect and the case of most interest is when $\mathscr{A}_{jilk} = O(kh\mathscr{A}_{jilk})$. In this case the correct traction continuity conditions are given by (53) and (54). With the use of (3) and (4), these two conditions can be re-written as

$$\alpha_{21}(u_{2,1} + u_{1,2}) = \frac{h\lambda_1}{\lambda_2}(W_{11} - W_{12}^2/W_{22})u_{1,11}, \tag{99}$$

$$\alpha_{12}u_{1,1} + \alpha_{22}u_{2.2} = (\frac{h}{\lambda_2})W_1 u_{2,11}, \tag{100}$$

where we have un-scaled u_1, u_2, x_1, x_2 by k and the moduli for the coating by kh. Assuming that solving $\sigma_2 = J^{-1}\lambda_2 W_2(\lambda,\ \lambda_2,\ 1) = 0$ yields $\lambda_2 = \lambda_2(\lambda)$ so that $W_2\ (\lambda,\ \lambda_2(\lambda),\ 1) \equiv 0$, we define a function $B(\lambda)$ through

$$B(\lambda) = \frac{h}{\lambda_2}W(\lambda,\ \lambda_2(\lambda),\ 1).$$

We note that the factor h/λ_2 in the above definition is the thickness of the coating in its unstressed configuration. It can then be shown that Eqs. (99) and (100) take the form

$$\alpha_{21}(u_{2,1} + u_{1,2}) = \lambda B_{\lambda\lambda}u_{1,11}, \tag{101}$$

$$\alpha_{12}u_{1,1} + \alpha_{22}u_{2,2} = B_\lambda u_{2,11}, \tag{102}$$

where subscript λ signifies differentiation with respect to λ. In terms of $B(\lambda)$, the constants s_1^* and s_2^* in the corresponding leading order bifurcation condition (27) are given by

$$s_1^* = k\lambda B_{\lambda\lambda}, \quad s_2^* = kB_{\lambda}. \tag{103}$$

Eqs. (101) and (102) are almost identical to Steigmann and Ogden's (1997) Eq. (6.19), the only difference being the extra term $-\lambda B_{\kappa\kappa}v''''_2$ in their Eq. (6.19a). This extra term gives rise to the extra term $k^3\lambda B_{\kappa\kappa}$ in their expression (6.28a) for a (with s_1^* and s_2^* replaced by a and b, respectively, our bifurcation condition (27) has the same form as their bifurcation condition (6.30)). This extra term reflects the fact that in Steigmann and Ogden's (1997) theory the thin plate does not behave like a membrane but instead it is endowed with a finite flexural stiffness by allowing B to depend on the curvature κ explicitly.

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